Meeting 10 — Math 22

Activity 10.6.5
- Complete w/ your group.
- Class discussion.

Activity 10.6.5. Consider the function f defined by $f(x,y) = -x + 2xy - y$.

a. Find the gradient $\nabla f(1,2)$ and sketch it on Figure 10.6.5.

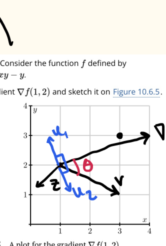


Figure 10.6.5. A plot for the gradient $\nabla f(1,2)$.

b. Sketch the unit vector $\mathbf{z} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ on Figure 10.6.5 with its tail at $(1,2)$. Now find the directional derivative $D_{\mathbf{z}}f(1,2)$.

c. What is the slope of the graph of f in the direction $\mathbf{z}$? What does the sign of the directional derivative tell you?

d. Consider the vector $\mathbf{v} = \langle 2, -1 \rangle$ and sketch $\mathbf{v}$ on Figure 10.6.5 with its tail at $(1,2)$. Find a unit vector $\mathbf{w}$ pointing in the same direction of $\mathbf{v}$. Without computing $D_{\mathbf{w}}f(1,2)$, what do you know about the sign of this directional derivative? Now verify your observation by computing $D_{\mathbf{w}}f(1,2)$.

e. In which direction (that is, for what unit vector $\mathbf{u}$) is $D_{\mathbf{u}}f(1,2)$ the greatest? What is the slope of the graph in this direction?

f. Corresponding, in which direction is $D_{\mathbf{u}}f(1,2)$ least? What is the slope of the graph in this direction?

g. Sketch two unit vectors $\mathbf{u}$ for which $D_{\mathbf{u}}f(1,2) = 0$ and then find component representations of these vectors.

h. Suppose you are standing at the point $(3,3)$. In which direction should you move to cause f to increase as rapidly as possible? At what rate does f increase in this direction?

(a) $\nabla f(x,y) = \langle -1+2y, 2x-1 \rangle$

$\nabla f(1,2) = \langle 3, 1 \rangle$

(b) $D_z f(1,2) = \nabla f(1,2) \cdot z = -\frac{4}{\sqrt{2}}$

(c) Slope in the z-direction is $-\frac{4}{\sqrt{2}}$. Since it's negative, f is decreasing in the direction of the vector z.

(d) $w = \frac{v}{|v|} = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$

We know $D_w f(1,2) > 0$ since θ is acute.

$D_w f(1,2) = \nabla f(1,2) \cdot w = \frac{6}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} > 0$

(e) $D_u f(1,2)$ is maximized when $u = \frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

The slope in this direction is $D_u f(1,2) = |\nabla f(1,2)| = \sqrt{10}$

(f) $D_u f(1,2)$ is minimized when $u = -\frac{\nabla f(1,2)}{|\nabla f(1,2)|}$

In this direction, the slope is $D_u f(1,2) = -\sqrt{10}$

(g) See similar problem in weekly 1

(h) The direction of greatest increase is $\nabla f(3,3)$. The rate of increase in this direction is $|\nabla f(3,3)|$.

Important Facts about ∇f
- $\nabla f(x,y)$ points in the direction of steepest ascent at (x,y).
- $-\nabla f$ points in the direction of steepest descent
- The rate of increase in the direction of steepest ascent $\left(u = \frac{\nabla f}{|\nabla f|}\right)$ at the point (x,y) is $D_u f(x,y) = |\nabla f(x,y)|$.
- The rate of decrease in the direction of steepest descent $\left(u = \frac{\nabla f}{|\nabla f|}\right)$ is $D_u f = |\nabla f|$
- The gradient $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = k$ where $k = f(a,b)$.

End of Section 10.6

Reading Debrief
- Discuss Activity 10.7.2 w/ your group
- Questions from Sections 10.7.1 - 10.7.2?

Section 10.7.2 — Second-Derivative Test

The Second Derivative Test.

Suppose (x_0, y_0) is a critical point of the function f for which $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$. Let D be the quantity defined by

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2.$$

1. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a local maximum at (x_0,y_0).
2. If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a local minimum at (x_0,y_0).
3. If $D < 0$, then f has a saddle point at (x_0,y_0).
4. If $D = 0$, then this test yields no information about what happens at (x_0,y_0).

The quantity D is called the *discriminant* of the function f at (x_0,y_0).

The discriminant D is just the determinant of the Hessian matrix

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f & f_{yy} \end{bmatrix}.$$

The values of $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ at the point (x_0,y_0) determine a Taylor polynomial of degree 2 that approximates the function f at (x_0,y_0).

The discriminant measures the curvature of the Taylor polynomial. The conditions in the 2nd Derivative test tell us what the Taylor polynomial looks like.

Case 1 $\rightsquigarrow$ Case 3 

Case 2 $\rightsquigarrow$ Case 4 Anything can happen

Since the Taylor polynomial approximates the function near (x_0,y_0), the cases tell us whether $f(x,y)$ has a local max/min or saddle point.

Activity 10.7.4
- Complete w/ the person next to you.
- Class discussion.

Activity 10.7.4. Find the critical points of the following functions and use the Second Derivative Test to classify the critical points.

a. $f(x,y) = 3x^3 + y^2 - 9x + 4y$

b. $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

c. $f(x,y) = x^3 + y^3 - 3xy$.

(a) $\nabla f(x,y) = \langle 9x^2 - 9, 2y + 4 \rangle = \langle 0, 0 \rangle$

Two critical points $(1,-2), (-1,-2)$.

$$D(x,y) = \det \begin{bmatrix} f_{xx} = 18x & f_{xy} = 0 \\ f_{yx} = 0 & f_{yy} = 2 \end{bmatrix} = 36x$$

$D(1,-2) = 36 > 0$ — $D(-1,-2) = -36 < 0$

$f_{xx}(1,-2) = 18 > 0$ — $(-1,2)$ is a saddle point

$(1,-2)$ Local min

(b) $\nabla f(x,y) = \langle y - \frac{2}{x^2}, x - \frac{4}{y^2} \rangle = 0$

$y = \frac{2}{x^2}$ $\quad x = \frac{4}{y^2} \Rightarrow x = \frac{4}{\left(\frac{2}{x^2}\right)^2} = x^4$

$\Rightarrow x = 1, y = 2$

There is one critical point $(1,2)$

$f_{xx} = \frac{4}{x^3}$ $\quad f_{xy} = 1$

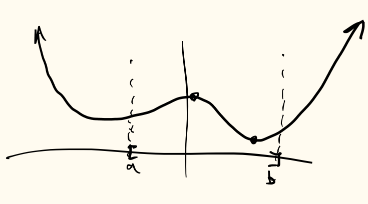
$f_{yx} = 1$ $\quad f_{yy} = \frac{8}{y^3}$

$$D(1,2) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = 3 > 0$$

$f_{xx}(1,2) = 4 > 0$ $\quad (1,2)$ is a local min

Section 10.7.3 — Optimization on a Restricted Domain

Extreme Value Theorem Let $f(x)$ which continuous on a closed interval $[a,b]$. Then f attains a global max and a global min on $[a,b]$.

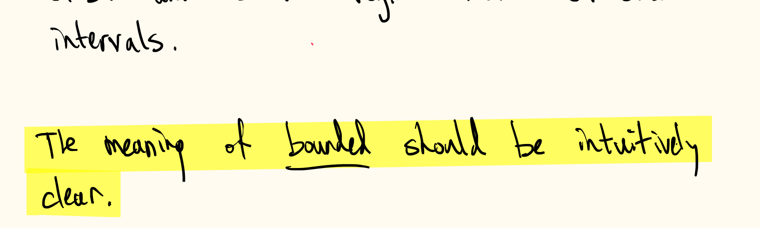


The absolute extrema occur at critical points $(f'(x) = 0)$ or at endpoints a and b.

In multivariable calc, we need to conside closed and bounded regions instead of closed intervals.

The meaning of bounded should be intuitively clear.

For example,



A closed region is a region which contains its boundary.

For example,
- The boundary of the open disk R_1 is the circle $(x-a)^2 + (y-a)^2 = r^2$. The open disk is not closed since it does not contain its boundary.
- The boundary of R_2 is the set of points on the x and y-axes. R_2 is closed because it contains its boundary.

Example

Extreme Value Theorem Let $f(x,y)$ be continuous on a closed and bounded region R. Then f attains a absolute max and min on R.

The absolute max and min occur either at critical points or on the boundary of R.

Example 10.7.8. Suppose the temperature T at each point on the circular plate $x^2 + y^2 \leq 1$ is given by

$$T(x,y) = 2x^2 + y^2 - y.$$

The domain $R = \{(x,y) : x^2 + y^2 \leq 1\}$ is a closed and bounded region, as shown on the left of Figure 10.7.9, so the Extreme Value Theorem assures us that T has an absolute maximum and minimum on the plate. The graph of T over its domain R is shown in Figure 10.7.9. We will find the hottest and coldest points on the plate.

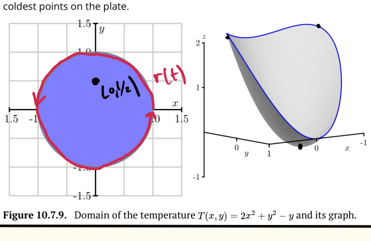


Figure 10.7.9. Domain of the temperature $T(x,y) = 2x^2 + y^2 - y$ and its graph.

Step 1: Find crit. points

$\langle 0,0 \rangle = \nabla f = \langle 4x, 2y - 1 \rangle$

One critical point $(0, 1/2)$

Step 2: Parametrize the boundary.

$r(t) = \langle \cos(t), \sin(t) \rangle$

$0 \leq t \leq 2\pi$

Step 3. Optimize the function $f(x(t), y(t))$. (calc I)

Find the critical points (including endpoints of interval)

$$0 = \frac{d}{dt} f(x(t), y(t))$$

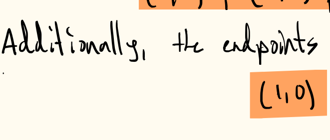
$$\text{chain rule} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= -4\cos(t)\sin(t) + (2\sin(t) - 1)\cos(t)$$

$$= \cos(t)(-4\sin t + 2\sin t - 1)$$

$$= \cos(t)(-2\sin t - 1)$$

Either $\cos(t) = 0$ or $\sin(t) = -\frac{1}{2}$



Four critical points on the boundary

$t = \pi/2, 3\pi/2, t = 7\pi/6, 11\pi/6$ which correspond to

$(0,1), (0,-1), (-\sqrt{3}/2, -1/2), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Additionally, the endpoints of $[0, 2\pi]$ give $(1,0)$

Step 4: Evaluate $f(x,y)$ at all critical points from preceding steps

$T(0, 1/2) = -1/4$ $\quad T(0,0) = 0$

$T(0,1) = 0$ $\quad T\left(-\frac{\sqrt{3}}{2}, -1/2\right) = \frac{9}{4}$

$T(0,-1) = 2$ $\quad T\left(\frac{\sqrt{3}}{2}, -1/2\right) = \frac{9}{4}$

Step 5. Identify the largest and smallest values

Global max $= \frac{9}{4}$ occurs at two points

Global min $= -\frac{1}{4}$ occurs at $(0, 1/2)$.